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Numerical Solutions of Some Linear Volterra Integral Equations Using Legendre, Chebyshev and Gegenbauer Polynomials

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ABSTRACT

In this work, the three outstanding polynomials- Legendre, Chebyshev and Gegenbauer polynomials and the well-known Galerkin method are used to obtain the approximate solutions of linear Volterra integral equations of both first and second kind including Abel's integral equation.

Keywords: Volterra equation; Legendre, Chebyshev and Gegenbauer

INTRODUCTION

Linear and nonlinear Volterra integral equations (VIE) of the first or second kind especially Abel's integral equation encompasses many significant mathematical approaches in modeling endeavor which spread over multiple variety of basic scientific disciplines , engineering disciplines and other connected areas. There are many works on different analytic methods namely series solution method, successive substitution method, Adomian's decomposition method and Laplace transform method which generally produce the analytic closed form solution of such equations. But, it is very important to employ numerical techniques to find solutions of many integral equations because of unavailability of their solutions in closed form.

Some classes of integral equations of both first and second kinds were solved numerically using Bernstein polynomials by Mandal and Bhattacharya [7] and Maleknejad et al [8] also approached same polynomial method for finding approximate solutions of some Volterra integral equations. To find numerical solutions of different Volterra integral equations, Rahman and Islam [10] used Legendre polynomials, Rahman and Islam [9] used Hermite and Chebyshev Polynomials and presented comparison, Shahsavaran[12] used Block-Pulse Functions and Taylor Expansion by Collocation Method.

In this paper, we have solved six illustrative examples of Volterra integral equations of first and second kind numerically by the technique of very well-known Galerkin method [5] and Legendre, Chebyshev and Gegenbauer piecewise polynomials [1] are used as trial function in the basis. Comparative performance of these polynomials in this regard of solving different varieties of VIE is also presented graphically.

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THE GENERAL METHOD:

In this section, first we consider the Volterra integral equation (VIE) of the first kind [1, 11, 14], given by

$$\int_a^x K(x,t)u(t)dt = f(x), \quad a \le x \le b \; (1)$$

where $u(x)$ is the unknown function, to be determined, $K(x,t)$ is the kernel function, continuous or discontinuous and $f(x)$ being the known function satisfying $f(a)=0$ .Now we use the technique of Galerkin method, [5], to find an approximate solution $\tilde{u}(x)$. For this, we assume that

$$\tilde{u}(x) = \sum_{i=0}^{n} c_i N_i(x) \; \ldots\ldots\ldots\ldots..(2)$$

where $N_i(x)$'s are Legendre or Chebyshev or Gegenbauer polynomials of degree i defined in equation by next section, c_i 's are unknown parameters to be determined and n is the number of piecewise polynomials. An approximate solution $\tilde{u}(x)$ will not produce an identically zero function but a function called the residual function. We get the residual function as

$$R(x) = \sum_{i=0}^{n} c_i \int_a^x K(x,t)N_i(t)dt - f(x), \quad a \le x \le b \; \ldots\ldots\ldots..(3)$$

Now the Galerkin equations corresponding to the approximation (2), given by

$$\int_a^x R(x)N_j(t)dx = 0 \; \ldots\ldots\ldots\ldots(4)$$

Using (3) and (4) after minor simplification, we obtain

$$\sum_{i=0}^{n} c_i \int_a^b \left[\int_a^x K(x,t)N_i(t)dt \right] N_j(x)dx = \int_a^b N_j(x) f(x)dx, \; j = 0,1,2,3,....n \; \ldots\ldots(5)$$

The above equations (5) are equivalent to the matrix form

$$DC = B \qquad \ldots\ldots\ldots..(6)$$

where the elements of the matrix C, D and B are and $c_i, d_{i,j}$ and b_j respectively, given by

$$c_i = [c_1, c_2, c_3, c_4, \ldots\ldots\ldots. c_n]^T$$

$$d_{i,j} = \int_a^b \left[\int_a^x K(x,t)N_i(t)dt \right] N_j(x)dx, \quad i,j = 0,1,2,3,\ldots\ldots n$$

$$b_j = \int_a^b N_j(x) f(x)dx, \; j = 0,1,2,3,....n \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots.(7)$$

Now the unknown parameters c_i are determined by solving the system of equations (7) and substituting these values of parameters in (2), we get the approximate solution $\tilde{u}(x)$ of the integral equation (1).

Now, we consider the Volterra integral equation (VIE) of the second kind [1, 11, 14] given by

$$u(x)+\lambda\int_a^x K(x,t)u(t)dt = f(x), \quad a \le x \le b \quad \ldots\ldots\ldots\ldots.. (8)$$

where $u(x)$ is the unknown function to be determined, $K(x,t)$ is the kernel function, continuous or discontinuous, $f(x)$ being the known function and λ is the constant. Then applying the same procedure as described above, we obtain the matrix form

$$DC = B \quad \ldots\ldots\ldots\ldots (9)$$

where the elements of the matrix C,D and B are and $c_i, d_{i,j}$ and b_j respectively, given by

$$c_i = [c_1, c_2, c_3, c_4, \ldots\ldots\ldots c_n]^T$$

$$d_{i,j} = \int_a^b \left[\int_a^x K(x,t)N_i(t)dt\right] N_j(x)dx, \; i,j = 0,1,2,3,\ldots\ldots n$$

$$b_j = \int_a^b N_j(x) f(x)dx, \; j = 0,1,2,3,\ldots.n \quad \ldots\ldots\ldots\ldots\ldots.(10)$$

Now the unknown parameters c_i are determined by solving the system of equations (10) and substituting these values of parameters in (2), we get the approximate solution $\tilde{u}(x)$ of the integral equation (8). The absolute error for this formulation is defined by Absolute Error $= |u(x)-\tilde{u}(x)|$.

The formulation for nonlinear integral equation will be discussed by considering numerical problems in the next section.

THE POLYNOMIAL BASES

Legendre Polynomials: The general form of the Legendre polynomials [10] of n-th degree is defined by

$$P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (2n-2r)}{2^n (r)(n-r)(n-2r)} x^{2n-r} \; ; \left[\frac{n}{2}\right] = \begin{cases} n/2 & if \;\; n \;\; is \;\; even \\ (n+1)/2 & if \;\; n \;\; is \;\; odd \end{cases}$$

Chebyshev Polynomials:

The Chebyshev polynomials, named after Pafnuty Chebyshev, are a sequence of orthogonal polynomials which are related to de Moivre's formula and which can be defined recursively. The general form of the Chebyshev polynomials [1] of nth degree is defined by

$$P_n(x) = \sum_{r=0}^{[n/2]} \frac{(-1)^r (n)}{2^n (2r)(n-2r)} (1-x^2)^r x^{n-2r} \; ; \left[\frac{n}{2}\right] = \begin{cases} n/2 & if \;\; n \;\; is \;\; even \\ (n+1)/2 & if \;\; n \;\; is \;\; odd \end{cases}$$

Gegenbauer Polynomials: The general form of the Gegenbauer polynomials [1] of n-th degree is defined by

$$C_n^1(x) = \sum_{r=0}^{\left[\frac{n}{2}\right]} \frac{(-1)^r \Gamma(n-r+1)}{r!(n-2r)!}(2x)^{n-2r} \; ; \left[\frac{n}{2}\right] = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

ILLUSTRATIVE EXAMPLES

Here we illustrate the above mentioned methods with the help of six illustrative examples, which include three first kind and three second kind Volterra integral equations with two regular kernels and four with weakly singular kernels.

Example 1: Consider the first kind Abel's integral equation [9]

$$\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt = \frac{2}{105}\sqrt{x}(105 - 56x^2 + 48x^3), \; 0 \le x \le 1 \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots \; (11)$$

The exact solution is $u(x) = x^3 - x^2 + 1$ …………………………… (12)

Using Legendre, Chebyshev and Gegenbauer polynomials and the formula derived in the equation (11) for n=10, we get the approximate solution is $u(x) = x^3 - x^2 + 1$, which is the exact solution although we obtained errors in the order of 10^{-16} for Legendre, Chebyshev and Gegenbauer polynomial before using the code for simplifying the approximate polynomial solution in Mathematica. . On the contrary, the accuracy is found nearly the order of 10^{-7} for $n = 10$ by using Bernstein approximation [2].

Example 2: Consider an Abel's integral equation (VIE of first kind with weakly singular kernels) of the form [9] $\int_0^x \frac{1}{\sqrt{x-t}} u(t)dt = x^5, \; 0 \le x \le 1$ ……………………… (13)

The exact solution is, $u(x) = \frac{1280}{315\pi} x^{\frac{9}{2}}$

Results found using the aforementioned three polynomial bases have been shown in Table 1 for $n = 10$. Graphs of the absolute errors among the approximate solutions found by using the polynomials for different values of x are portrayed in Figure 1.1 for $n = 10$. The absolute errors are obtained in the order of 10^{-11}, 10^{-8} and 10^{-10} for Legendre, Chebyshev and Gegenbauer polynomials basis respectively for $n = 10$. On the other hand, the absolute errors were obtained by Mandal and Bhattacharya [7] in the order of 10^{-7} for $n = 10$ on their case of Bernstein's polynomials. Also, by using Hermite and Chebyshev polynomials [2], the absolute errors were obtained in the order of 10^{-8} for $n = 10$.

Example 3: We consider the Volterra integral equation of the first kind of the form [7]

$\int_0^x \frac{1}{x^2+t^2} u(t)dt = x$, $0 \le x \le 1$ ……… (14) which has the exact solution, $u(x) = \frac{4}{4-\pi} x^2$

Table 1: Value X

Value x	Exact Value	Error Legendre	Error Chebyshev	Error Gegenbauer
0.0	0	5.13597390111145E-9	7.81382957397230E-8	1.035634223484429E-8
0.1	0.00004090247078	4.43825922579472E-10	1.79162573994305E-8	1.746737563544090E-9
0.2	0.00092551726280	3.28625681366635E-10	6.61708506884779E-9	1.076778299993360E-9
0.3	0.00573845776258	2.01759171598012E-10	1.43124477625718E-8	1.523428222472115E-9
0.4	0.02094206504431	4.10407071151808E-11	1.39012686079230E-8	1.273204309521177E-9
0.5	0.05716294070838	5.50445769511818E-11	1.08402477386643E-8	8.638695742836547E-10
0.6	0.12984647671933	9.42344310621103E-11	5.15487728891539E-9	2.606089144411648E-10
0.7	0.25983085548789	5.39638170007387E-11	5.44123377859686E-9	7.346539900731727E-10
0.8	0.47386483855632	1.01587793593791E-10	1.83899482716833E-8	1.776374311619803E-9
0.9	0.80508333255323	3.26004586629304E-11	1.15052754070556E-8	1.264312650560714E-9
1.0	1.29344969623889	2.55513633730427E-9	2.78497938403739E-7	2.636737068141797E-8

Results found using the aforementioned three polynomial bases have been shown in Table 2 for $n=10$. Graphs of the absolute errors between the exact solution and the approximate solution for different values of x are portrayed in Figures 1.2 for $n=10$. The absolute errors are obtained in the order 10^{-11} , 10^{-8} and 10^{-9} for Legendre, Chebyshev and Gegenbauer polynomials basis respectively for $n=10$. On the other hand, the absolute errors were obtained in the order of 10^{-12} for $n=10$ (degree of Bernstein's polynomials) by Bhattacharya and Mandal [7].

Table 2: Value X_1

Value x	Exact Value	Error Legendre	Error Chebyshev	Error Gegenbauer
0.0	0	1.27861313965214E-12	1.4417787391762E-10	4.33772327613850E-11
0.1	0.04659792366325	1.89298114113296E-11	2.5031247423255E-9	7.43061711140994E-10
0.2	0.18639169465301	2.06057217200480E-11	3.1641265852336E-9	9.30351318292904E-10
0.3	0.41938131296929	3.43867868111772E-11	3.3568217172808E-9	1.02026983529668E-9
0.4	0.74556677861207	1.05199269921296E-10	1.3516609234076E-8	4.02104945439717E-9
0.5	1.16494809158137	6.55651116159591E-11	1.0457856356033E-8	3.07206169732431E-9
0.6	1.67752525187717	1.01713009261525E-10	1.1727520603578E-8	3.51393643001808E-9
0.7	2.28329825949948	2.32712406723860E-10	3.2398446831591E-8	9.59242368175143E-9
0.8	2.98226711444831	2.17056879925115E-10	3.3471384603789E-8	9.85681341612869E-9
0.9	3.77443181672364	2.50013223326747E-10	4.0316383345011E-8	1.18482671701455E-8
1.0	4.65979236632548	6.30774094633836E-9	9.0915954153109E-7	2.68739644501826E-7

Example 4: Consider the second kind Abel's integral equation [9] of the form

$$u(x)-\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt = x^7(1-\frac{4096}{6435}\sqrt{x}),\ 0\le x\le 1 \quad \text{……………..} (15)$$

which has the exact solution, $u(x)=x^7$. Using Legendre, Chebyshev and Gegenbauer polynomials and the formula derived in the equation (11) for $n=10$, we get the approximate solution is $u(x)=x^7$, which is the exact solution although we obtained errors in the order of 10^{-17} before using the code for simplifying the approximate polynomial solution Mathematica. On the other hand, the absolute errors were obtained in the order of 10^{-7} for $n=10$ (degree of Bernstein's polynomials) by Bhattacharya and Mandal [7]

Example 5: Consider the second kind Abel's integral equation [4] of the form

$$u(x)+\frac{1}{4}\int_0^x \frac{1}{\sqrt{x-t}}u(t)dt = \frac{1}{\sqrt{x+1}}+\frac{\pi}{8}-\frac{1}{4}\sin^{-1}\frac{1-x}{1+x},\ 0\le x\le 1 \quad \text{………………}(16)$$

the exact solution $u(x)=\frac{1}{\sqrt{x+1}}$

Results found using the aforementioned three polynomial bases have been shown in Table 3 for $n=10$. Graphs of the absolute errors between the exact solution and the approximate solution for different values of x are portrayed in Figures 2.1 for $n=10$. The absolute errors are obtained in the order of 10^{-9} , 10^{-9} and 10^{-9} for Legendre, Chebyshev and Gegenbauer polynomial basis respectively for $n = 10$. On the other hand, the absolute errors were obtained in the order of 10^{-3} [12] reported the error up to for $K = 64$ (number of block pulse function)

Table 3 : Value X_2

Value x	Exact Value	Error Legendre	Error Chebyshev	Error Gegenbauer
0.0	1	9.59502803384918E-9	1.01251286354447E-8	5.20723530695421E-9
0.1	0.95346258924559	2.70299934513793E-9	2.86652737921985E-9	1.50179749459054E-9
0.2	0.91287092917527	1.99607728219855E-9	2.12932481972419E-9	1.10852423051565E-9
0.3	0.87705801930702	1.03849816434828E-10	5.21426820144566E-11	3.20934854147751E-10
0.4	0.84515425472851	2.46681995421988E-9	2.53896954981479E-9	1.86555547193989E-9
0.5	0.81649658092772	2.81552714289900E-9	2.94824872332305E-9	1.94561010309576E-9
0.6	0.79056941504209	6.27414473476345E-10	6.93704128542204E-10	2.91286195480557E-10
0.7	0.76696498884737	2.00786360977645E-9	2.06197284705736E-9	1.58107235671783E-9
0.8	0.74535599249992	3.32255431723152E-9	3.45185235908552E-9	2.47888841724339E-9
0.9	0.72547625011001	3.91502149887979E-9	4.07280244446011E-9	2.91971657998331E-9
1.0	0.70710678118654	1.30022160284892E-8	1.35368096186118E-8	9.713976892743897E-9

Example 6: Consider the second kind Abel's integral equation [14] of the form

$$u(x) + \int_0^x (x-t)u(t)dt = x, \quad 0 \le x \le 1 \quad \ldots\ldots\ldots\ldots\ldots (17)$$

the exact solution, $u(x) = \sin x$

Results found using the aforementioned three polynomial bases have been shown in Table 2.2 for $n = 10$. Graphs of the absolute errors between the exact solution and the approximate solution for different values of x are portrayed in Figures 2.2 for $n = 10$.

Table 4: ValueX_3

Value x	Exact Value	Error Legender	Error Chebyshev	Error Gegenbauer
0.0	0	3.13488209448354E-14	3.13488209675312E-14	3.13488209236389E-14
0.1	0.09983341664682	8.97892871165595E-15	8.95117313604032E-15	8.97892871165595E-15
0.2	0.19866933079506	7.38298311375729E-15	7.41073868937292E-15	7.38298311375729E-15
0.3	0.29552020666133	7.60502771868232E-15	7.66053886991358E-15	7.60502771868232E-15
0.4	0.38941834230865	5.49560397189452E-15	5.55111512312578E-15	5.55111512312578E-15
0.5	0.47942553860420	0	5.55111512312578E-17	0
0.6	0.56464247339503	5.44009282066326E-15	5.21804821573823E-15	5.32907051820075E-15
0.7	0.64421768723769	7.54951656745106E-15	7.77156117237609E-15	7.43849426498854E-15
0.8	0.71735609089952	7.32747196252603E-15	7.32747196252603E-15	7.21644966006351E-15
0.9	0.78332690962748	8.65973959207622E-15	8.88178419700125E-15	8.88178419700125E-15
1.0	0.84147098480789	3.06421554796543E-14	3.06421554796543E-14	3.06421554796543E-14

CONCLUSION

Our attempt to finding solutions of chosen all six VIE by applying outstanding Galerkin method using the three polynomials as trial basis was led to a quite satisfactory standpoint. Clearly, all our presented examples of different variety encompassing linear, nonlinear; first kind, second kind; regular kernel, weakly singular kernel are strongly in concordance with the exact solutions. The solutions of two examples coincide with the exact solutions for the case of all the three polynomials which caught our attention in developing the rigorous assumption that the approximate solution using polynomial basis by above-followed numerical method will always produce identical solution for any VIE involving polynomial type exact solution. We came to perceive that as per performance of reaching the numerical solution to the closeness of the exact solution, of the three polynomials Legendre polynomial was best, Gaugenbaur was better and Chevyshev was good. We believe that these rigorous polynomials would produce surprisingly more accurate approximate solutions than currently found ones if technological limitation of choosing polynomials of lower degree could be avoided.

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